

Buddy Allocator

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1 Average Internal Fragmentation in Buddy Allocator

In a buddy system, memory is managed in blocks of size 2^k . When a request comes in for size s , the smallest block 2^k is allocated such that: $2^k \geq s$.

- This means the internal fragmentation is: $IF(s) = \frac{2^k - s}{2^k}$

1.1 Minimum Fragmentation

If $s = 2^k$, fragmentation is 0%.

1.2 Maximum Fragmentation

If $s = 2^{k-1} + 1$, it approaches close to 50% as k increases. Fragmentation can get arbitrarily close to 50%, but never quite hit it (since $s > 2^{k-1}$).

$$\lim_{x \rightarrow \infty} \frac{2^{k-1} + 1}{2^k} = \lim_{x \rightarrow \infty} \left[\frac{2^{k-1}}{2^k} + \frac{1}{2^k} \right] = \frac{1}{2} + \lim_{x \rightarrow \infty} \frac{1}{2^k} = \frac{1}{2} + 0 = 0.5 \quad (1)$$

1.3 Proof

- Assume that request sizes are uniformly distributed over the range: $s \in (2^{k-1}, 2^k]$ for $k \geq 1$.
- This means that the request size always falls into the range where the next larger block 2^k will be allocated.
- So, internal fragmentation becomes: $IF(s) = \frac{2^k - s}{2^k}$
- The probability density function for the uniform distribution would be $f(s) = \frac{1}{2^k - 2^{k-1}} = \frac{1}{2^{k-1}}$
- Then the expected value of internal fragmentation:

$$E[IF] = \int_{2^{k-1}}^{2^k} \frac{2^k - s}{2^k} \cdot \frac{1}{2^{k-1}} \delta s = \frac{1}{2^k \cdot 2^{k-1}} \int_{2^{k-1}}^{2^k} (2^k - s) \delta s \quad (2)$$

$$\frac{1}{2^k \cdot 2^{k-1}} \int_{2^{k-1}}^{2^k} (2^k - s) \delta s = \frac{1}{2^k \cdot 2^{k-1}} \left[2^k s - \frac{s^2}{2} \right]_{2^{k-1}}^{2^k} \quad (3)$$

$$\frac{1}{2^k \cdot 2^{k-1}} \left[2^k s - \frac{s^2}{2} \right]_{2^{k-1}}^{2^k} = \frac{1}{2^k \cdot 2^{k-1}} [(2^{2k} - 2^{2k-1}) - (2^{2k-1} - 2^{2k-3})] \quad (4)$$

$$\frac{1}{2^k \cdot 2^{k-1}} [(2^{2k} - 2^{2k-1}) - (2^{2k-1} - 2^{2k-3})] = \frac{1}{2^k \cdot 2^{k-1}} [2^{2k-1} - 2^{2k-1} + 2^{2k-3}] \quad (5)$$

$$\frac{1}{2^k \cdot 2^{k-1}} [2^{2k-1} - 2^{2k-1} + 2^{2k-3}] = \frac{2^{2k-3}}{2^k \cdot 2^{k-1}} \quad (6)$$

$$\frac{2^{2k-3}}{2^k \cdot 2^{k-1}} = \frac{2^{2k-3}}{2^{2k-1}} = \frac{1}{4} = 0.25 \quad (7)$$

- Hence, the average amount of internal fragmentation in Buddy Allocator is 25%.